ON THE STABILITY "IN THE LARGE" OF THE SOLUTIONS OF A DENUMERABLE SYSTEM OF DIFFERENTIAL EQUATIONS UNDER CONTINUOUSLY ACTING DISTURBANCES

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1. Statement of the problem. Let us consider a denumerable system of differential equations

$$\frac{dx_s}{dt} = \omega_s(t, x_1, x_2, \ldots) + f_s(t, x_1, x_2, \ldots) \qquad (s = 1, 2, \ldots) \qquad (1.1)$$

Here, t is a real independent variable; $x_1, x_2, \ldots$ are real unknown functions of t; $\omega_1, \omega_2, \ldots$ are given real functions of the quantities $t, x_1, x_2, \ldots$ in some region

$$t \geqslant 0, \quad \sup[|x_1|, |x_2|, \ldots] \leqslant L \qquad (1.2)$$

The functions $f_1, f_2, \ldots$, which in the sequel will be called disturbances, are in general unknown, but they satisfy within the region (1.2) the condition

$$|f_s(t, x_1, x_2, \ldots)| \leqslant \rho \qquad (s = 1, 2, \ldots) \qquad (1.3)$$

where $\rho > 0$ is some quantity determined in each case for the particular problem under consideration.

Let us assume that the right-hand sides of the system of Equations (1.1) satisfy within the region (1.2) the following conditions:

1) The functions ω_s and f_s are single-valued and continuous in t at each point of the region (1.2).

2) For any two points $(t, x_1', x_2', \ldots)$ and $(t, x_1'', x_2'', \ldots)$ we

have the inequality

$$|\omega_s(t, x_1'', x_2'', \ldots) - \omega_s(t, x_1', x_2', \ldots)| \leqslant A\Delta x \qquad (s = 1, 2, \ldots) \quad (1.4)$$

where A is some constant, and $\Delta x = \sup\,[|x_1'' - x_2'|,\ |x_2'' - x_1'|,\ \ldots]$. The disturbances f_s, too, satisfy the inequality (1.4).

It is known [1] that under the conditions imposed on the right-hand sides of the system of Equations (1.1), this system admits a solution, and every equicontinuous solution which passes through a given point $(t_1, x_1, x_2, \ldots)$ of the region (1.2) will be unique. A solution will exist, at least, for all those values $t \geqslant t_0 \geqslant 0$ for which its norm does not exceed a number L.

We will consider the following three regions, which we will call "rings":

$$t \geqslant 0, \qquad l_0 \leqslant \sup\,[|x_1|, |x_2|, \ldots] \leqslant l \qquad (1.5)$$

$$t \geqslant 0, \qquad l \leqslant \sup\,[|x_1|, |x_2|, \ldots] \leqslant L \qquad (1.6)$$

$$t \geqslant 0, \qquad l_0 \leqslant \sup\,[|x_1|, |x_2|, \ldots] \leqslant L \qquad (1.7)$$

Besides the system of Equations (1.1), we will consider the system of differential equations without disturbances

$$\frac{dx_s}{dt} = \omega_s(t, x_1, x_2, \ldots) \qquad (s = 1, 2, \ldots) \qquad (1.8)$$

Definition 1.1. A solution of the system of Equations (1.8) will be said to be stable "in the large" under constantly acting disturbances, if for any initial value $t = t_0 \geqslant 0$ there exists a number $\rho = \rho(t_0) > 0$ such that every solution of the system of Equations (1.1), which passes through a point of the region

$$\sup\,[|x_1|, |x_2|, \ldots] \leqslant l$$

will satisfy the inequality

$$\sup\,[|x_1|, |x_2|, \ldots] < L$$

for all $t > t_0$, and for arbitrary disturbances satisfying (1.3). In the opposite case we will say that the solutions of the Equations (1.8) are unstable "in the large" under constantly acting disturbances.

If the solution is stable, and if the number $\rho > 0$ can be selected independently of $t_0 > 0$, then this solution of the system (1.8) will be called uniformly stable "in the large" under constantly acting disturbances.

Definition 1.2. We shall say that the solutions of the system of

Equations (1.8) have total, strong instability "in the large" under constantly acting disturbances if for every $\rho > 0$ the solution of the Equations (1.1) that passes through an arbitrarily chosen point $(t_0, x_1, x_2, \ldots)$, and satisfies the condition

$$\sup [\,|x_{10}|,\ |x_{20}|, \ldots] = l$$

will satisfy the inequality

$$\sup [\,|x_1|,\ |x_2|, \ldots] \geqslant L$$

under arbitrary disturbances and for some $t > t_0$.

2. Some theorems. We will consider real functions $V(t, x_1, x_2, \ldots)$ of the variables $t, x_1, x_2, \ldots$. The functions are assumed to be defined, and continuous in the ring (1.7).

Definition 2.1. The function $V(t, x_1, x_2, \ldots)$ is said to be of constant sign if the inequality

$$V(t, x_1, x_2, \ldots) \geqslant 0 \quad (\text{ or } V(t, x_1, x_2, \ldots) \leqslant 0)$$

is satisfied in the ring (1.7).

Definition 2.2. A function $V(t, x_1, x_2, \ldots)$ that is of a constant sign in the ring (1.7) will be said to be sign-definite if its $\sup [|V(t, x_1, x_2, \ldots)|]$, for $t \geqslant 0$, on the surface $\sup [|x_1|, |x_2|, \ldots] = l$, is not greater than its $\inf [|V(t, x_1, x_2, \ldots)|]$ on the surface $\sup [|x_1|, |x_2|, \ldots] = L$.

Definition 2.3. A function $V(t, x_1, x_2, \ldots)$, that is of constant sign, will be said to be totally sign-definite if, in the ring (1.7), the inequality

$$V(t, x_1, x_2, \ldots) \geqslant \beta \quad (\text{ or } -V(t, x_1, x_2, \ldots) \geqslant \beta)$$

is valid when β is some constant.

We will assume that the function $V(t, x_1, x_2, \ldots)$ satisfies, in the region (1.2), the Lipschitz condition

$$|V(t + \Delta t, x_1'', x_2'', \ldots) - V(t, x_1', x_2', \ldots)| \leqslant k(|\Delta t| + \Delta x) \qquad (2.1)$$

where $k > 0$ is some constant, while $\Delta x = \sup [|x_1'' - x_1'|, |x_2'' - x_2'|, \ldots]$. It follows from [2] that the function $V(t, x_1, x_2, \ldots)$ has, in the region (1.2) along any integral curve of the system (1.1), or of the system (1.8), a total derivative in t (for almost all values of t). We will denote this derivative by dV/dt, or by V'. These will, in turn, reproduce the original function, i.e. we have the equations

$$V = V(t_0, x_{10}, x_{20}, \dots) + \int_{t_0}^{t} \frac{dV}{dt}\, d\tau, \qquad V = V(t_0, x_{10}, x_{20}, \dots) + \int_{t_0}^{t} V'\, d\tau$$

where the integrals are taken in the sense of Lebesgue. For the sake of definiteness, one may consider the expressions

$$\left(\frac{dV}{dt}\right)_+, \quad \left(\frac{dV}{dt}\right)_-, \quad V_+', \quad V_-'$$

(which exist at every point of the integral curve, and are equal for almost all values of t) as dV/dt, and V'.

Theorem 2.1. Let $V(t, x_1, x_2, \dots)$ be a function that satisfies the condition (2.1) in the ring (1.6). Then the inequalities

$$\left|\frac{dV}{dt} - V_+'\right| \leqslant k\rho, \qquad \left|\frac{dV}{dt} - V_-'\right| \leqslant k\rho$$

will be valid in (1.6) at almost all points of any integral curve of the system of Equations (1.1).

Proof. Let $y_1, y_2, \dots$ be a solution of the equations without disturbances, that passes through an arbitrarily chosen point $(t, x_1, x_2, \dots)$ of the region (1.2), and let $z_1, z_2, \dots$ be a solution of the Equations (1.1) which passes through the same point $(t, x_1, x_2, \dots)$. Then we will have

$$z_s = x_s + \int_t^{t+\Delta t} \omega_s(\tau, z_1, z_2, \dots)\, d\tau + \int_t^{t+\Delta t} f_s(\tau, z_1, z_2, \dots)\, d\tau$$

$$y_s = x_s + \int_t^{t+\Delta t} \omega_s(\tau, y_1, y_2, \dots)\, d\tau \qquad (s = 1, 2, \dots)$$

Setting $w_s = z_s - y_s$ $(s = 1, 2, \dots)$, we obtain

$$w_s = \int_t^{t+\Delta t} [\omega_s(\tau, z_1, z_2, \dots) - \omega_s(\tau, y_1, y_2, \dots) + f_s(\tau, z_1, z_2, \dots)]\, d\tau$$

$$(s = 1, 2, \dots)$$

from which it follows that

$$|w_s| \leqslant \int_t^{t+\Delta t} A\,\|w\|\, d\tau + \rho\Delta t \qquad (s = 1, 2, \dots)$$

and, therefore

$$\|w\| \leqslant \int_{t}^{t+\Delta t} A\|w\|d\tau + \rho\Delta t \tag{2.2}$$

where we assume that $\Delta t > 0$, and where $|w| = \sup\,[|w_1|, |w_2|, \ldots]$.

Let u be a solution of the equation

$$u = \int_{t}^{t+\Delta t} Au\,d\tau + \rho\Delta t$$

Then it is obvious that $|w| \leqslant u$, i.e.

$$\|w\| \leqslant \frac{e^{A\Delta t} - 1}{A}\rho$$

Suppose that γ is an arbitrarily chosen integral curve of the system of Equations (1.1). Since dV/dt exists for almost all values of t on this integral curve, we may assume that dV/dt exists at a point $(t, x_1, x_2, \ldots)$. The expressions dV/dt and V_+' may be written in the form

$$\frac{dV}{dt} = \lim_{\Delta t \to 0} \frac{V(t+\Delta t, z_1, z_2, \ldots) - V(t, x_1, x_2, \ldots)}{\Delta t} \tag{2.3}$$

$$V_+' = \lim_{\Delta t \to 0} \frac{V(t+\Delta t, y_1, y_2, \ldots) - V(t, x_1, x_2, \ldots)}{\Delta t} \tag{2.4}$$

where we mean by the expression $\Delta t \to 0$ the set of all those values of $\Delta t \to 0$ for which the limit V_+' exists.

On the basis of the condition (2.1) and the inequality (2.2), we will have

$$|V(t+\Delta t, z_1, z_2, \ldots) - V(t+\Delta t, y_1, y_2, \ldots)| \leqslant k\|w\| \leqslant k\frac{e^{A\Delta t} - 1}{A}\rho$$

It follows then from (2.3) and (2.4) that

$$\left|\frac{dV}{dt} - V_+'\right| \leqslant k\rho$$

In an analogous way one can show that

$$\left|\frac{dV}{dt} - V_-'\right| \leqslant k\rho$$

Theorem 2.2. If the system of differential equations without disturbances is such that there exists, in the ring (1.6), a sign-definite function $V(t, x_1, x_2, \ldots)$, satisfying the Lipschitz condition (2.1), whose derivative $V'(t, x_1, x_2, \ldots)$, in view of these equations, is in the ring (1.6) a totally sign-definite function of opposite sign from $V(t, x_1, x_2, \ldots)$, then the solutions of the system will be stable in the large also under constantly acting disturbances.

Proof. For the sake of definiteness we will assume that the function $V(t, x_1, x_2, \dots)$ is positive definite in the ring (1.6). Under this condition, its derivative $V'(t, x_1, x_2, \dots)$ will be negative and a totally sign-definite function satisfying in the ring (1.6) the condition $-V'(t, x_1, x_2, \dots) \geqslant \beta$.

By Theorem 2.1 we then have

$$-\frac{dV}{dt} \geqslant -V' - k\rho \geqslant \beta - k\rho \geqslant \frac{\beta}{2}$$

provided the quantity ρ is chosen so that $\rho < \beta/2k$. Therefore, we have the inequality

$$\frac{dV}{dt} \leqslant -\frac{\beta}{2}$$

in the ring (1.6).

Let us suppose that the system of Equations (1.1) has a solution $y_s = y_s(t)$ $(s = 1, 2, \dots)$, which on some segment $[t', t'']$ satisfies the conditions

$$\sup[|y_1(t')|, |y_2(t')|, \dots] = l, \qquad \sup[|y_1(t'')|, |y_2(t'')|, \dots] = L$$

and on the interval (t', t'') the condition

$$l < \sup[|y_1(t)|, |y_2(t)|, \dots] < L$$

Then we have

$$V_{t''} = V_{t'} + \int_{t'}^{t''} \frac{dV}{dt}\, d\tau \leqslant V_{t'} - \frac{\beta}{2}(t'' - t') < V_{t'} \leqslant \gamma_l \tag{2.5}$$

$$V_{t''} \geqslant \gamma_L \tag{2.6}$$

Here,

$$\gamma_l = \sup\ [V(t, x_1, x_2, \dots)] \text{ when } \sup[|x_1|, |x_2|, \dots] = l$$
$$\gamma_L = \inf\ [V(t, x_1, x_2, \dots)] \text{ when } \sup[|x_1|, |x_2|, \dots] = L$$

Since by the hypothesis of the theorem $\gamma_l \leqslant \gamma_L$, the inequalities (2.5) and (2.6) are contradictory, and, hence, the assumption on the instability is not valid, i.e. we will have stability, and, as a matter of fact, uniform stability, as is easily seen.

Theorem 2.3. If the system of differential equations without disturbances is such that there exists a sign-definite function

$V(t, x_1, x_2, \ldots)$ which is bounded in the ring (1.7), and which satisfies in this ring the condition (2.1), while in the ring (1.5) this function is sign-definite, and its derivative $V'(t, x_1, x_2, \ldots)$ is a totally sign-definite function in (1.7) with the same sign as $V(t, x_1, x_2, \ldots)$, then the solutions of the system will have total, strong instability "in the large" under constantly acting disturbances.

Proof. For the sake of definiteness let us assume that the function $V(t, x_1, x_2, \ldots)$ is positive definite in the ring (1.5). Then its derivative $V'(t, x_1, x_2, \ldots)$ will be totally positive definite in the same ring.

By the hypothesis of the theorem, if we set

$$\gamma_{l_0} = \sup [V(t, x_1, x_2, \ldots] \text{ when } t \geqslant 0,\ \sup[|x_1|, |x_2|, \ldots] = l_0$$

then we will have the inequalities

$$\gamma_{l_0} \leqslant \gamma_l = \inf [V(t, x_1, x_2, \ldots)] \quad \text{when } \geqslant 0,\ \sup[|x_1|, |x_2|, \ldots] = l$$

$$V' \geqslant \beta \quad \text{when } t \geqslant 0,\ l_0 \leqslant \sup[|x_1|, |x_2|, \ldots] \leqslant L$$
$$V \leqslant M \quad \text{when } t \geqslant 0,\ l_0 \leqslant \sup[|x_1|, |x_2|, \ldots] \leqslant L$$

Suppose that

$$x_1 = y_1(t), \qquad x_2 = y_2(t), \ldots \tag{2.7}$$

is a solution of the system of Equations (1.1) which passes through an arbitrary point $(t_0, x_{10}, x_{20}, \ldots)$ satisfying the condition

$$\sup[|x_{10}|, |x_{20}|, \ldots] = l$$

Let us suppose that $[t_0, t_1]$ is a segment such that for all values of t satisfying the inequality $t_0 \leqslant t \leqslant t_1$ we have the inequality

$$l_0 \leqslant \sup[|y_1(t)|, |y_2(t)|, \ldots] \leqslant L$$

On the basis of the Theorem 2.1, we can see that for the segment $[t_0, t_1]$, along an integral curve of (1.1), the inequality $dV/dt \geqslant \beta/2$ will be valid for almost all t on this segment, provided $\rho \leqslant \beta/2k$.

If $t \in [t_0, t_1]$, then

$$V_t = V_{t_0} + \int_{t_0}^{t} \frac{dV}{dt} d\tau \geqslant V_{t_0} + \frac{\beta}{2}(t - t_0) > V_{t_0} \geqslant \gamma_l \geqslant \gamma_{l_0}$$

i.e.

$$V_t > V_{l_0} \tag{2.8}$$

Therefore, for all values $t \in [t_0, t_1]$, we have the inequality $\sup [|y_1(t)|, |y_2(t)|, \ldots] > l_0$. In the opposite case, there would exist a value $t > t_0$ such that $V_t \leqslant V_0$, which would contradict (2.8).

On the other hand, for the indicated value $t \in [t_0, t_1]$ we have the inequality

$$M \geqslant V \geqslant \gamma_{l_0} + \frac{\beta}{2}(t - t_0) \qquad (2.9)$$

Since the right-hand side of the inequality (2.9) cannot hold for values of t such that

$$t > 2\frac{M - \gamma_{l_0}}{\beta} + t_0$$

there must exist a value

$$t = t'' > t_0 \qquad \left(t'' < 2\frac{M - \gamma_{l_0}}{\beta} + t_0\right),$$

such that

$$\sup [\,|y_1(t'')|,\ |y_2(t'')|, \ldots] = L$$

This implies that the solutions of the system of Equations (1.8) possess total, strong instability "in the large" under constantly acting disturbances.

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